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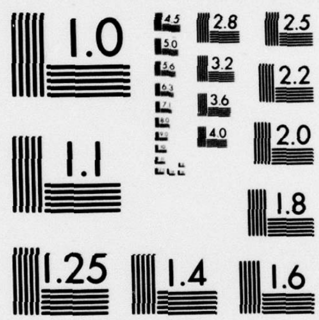
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NOTE ON THE EFFECT OF TRANSVERSE SHEAR DEFORMATION IN LAMINATED ANISOTROPIC PLATES,

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by E. Reissner

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IN LAMINATED ANISOTROPIC PLATES

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ABSTRACT

Derivation of two-dimensional equations for elastic deformations of laminated anisotropic plates, based on the assumption of a Kirchhoff-distribution of primary strains, in conjunction with the use of the Castigliano variational principle for stresses, or a variational principle for stresses and displacements. A discussion is given of the relation of the present work to some earlier work.

Note on the Effect of Transverse Shear Deformation in Laminated Anisotropic Plates†

By E. Reissner

Introduction. The recent work by Argyris et al. [1] and by G. A. Cohen [2] on the approximate determination of transverse shear stress coefficients for laminated anisotropic plates (and shells) has shown the relative complexity of the task of extending the earlier analysis of the effect of transverse shear in isotropic homogeneous [3] as well as sandwich-type plates [4] to the case of laminated anisotropic plates. The point of departure of the following considerations, which were motivated by Cohen's paper, is a brief note [6] on an earlier rational approach to the subject of transverse shear stiffness in laminated anisotropic plates, which, for brevity's sake, was limited to the case of plates with such symmetry as to allow a separate treatment of transverse bending and stretching.

A study of the contents of [2] made it appear that the results in this paper, although quite different in appearance and derivation, should, for

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the symmetrical case, be in fact equivalent to the results reported in [6]. This being the case, the present brief paper, although not presenting numerical results or procedures, is thought to be of some interest from the point of view of the development of computer methods in applied mechanics.

Briefly, the present work as well as the work in [2] and [6] starts out with a distribution of primary stresses in laminated plates corresponding to a linear (Kirchhoff) distribution of primary strains. The second step in the three papers is a determination of the associated statically consistent transverse shearing stresses. The remaining step is the use of an energy principle for the utilization of these expressions for primary and transverse stresses for the purpose of obtaining two-dimensional constitutive equations involving transverse shear deformations and transverse shear stress resultants. In [6] this third step is executed, not quite as simply as it might have been done, through an application of the writer's variational principle for stresses <u>and</u> displacements [5]. In Cohen's work [2] this third step is executed through an application of the principle of minimum complementary energy (variational principle for stresses). In the present reconsideration of the problem two types of results are presented. These are associated, respectively, with the variational principle for stresses, or with the variational principle for stresses and displacements. It is found that, while Cohen's <u>method</u> [2] depends on use of the stress principle, his <u>results</u> appear to be equivalent to the results in [6] which depend on the use of the stress and displacement principle.

Stress Strain Relations and Expressions for Stresses. We consider a plate with midplane coordinates x_1, x_2, and with thickness coordinate z. We designate midplane parallel stresses by σ_1, σ_2, σ_3 (with σ_3 representing the shear stress $\sigma_{12} = \sigma_{21}$), and we designate transverse shearing stresses by τ_1, τ_2. We assume that the deformational effects of transverse normal stress - which would here be designated by τ_3 - are negligible, and we assume that the relevant stress strain relations for the laminated anisotropic plate are of the form

$$\sigma_i = E_{ij} e_j \quad , \quad \tau_\lambda = G_{\lambda\mu} g_\mu \quad , \tag{1a, b}$$

with the coefficients E_{ij} and $G_{\lambda\mu}$ being known functions of z.

Given the stress strain relations (1a), we now introduce as a basic assumption for the analysis to be undertaken that the distribution of face-parallel stresses may be approximated in the form

$$\sigma_i = E_{ij}(z\varkappa_j + \epsilon_j) \quad , \tag{2}$$

with $\varkappa_j$ and ϵ_j being independent of z.

Equations (2) imply as relations for stress couples and stress resultants,

$$M_i = D_{ij}\varkappa_j + C_{ij}\epsilon_j \quad , \quad N_i = C_{ij}\varkappa_j + B_{ij}\epsilon_j \quad , \tag{3a, b}$$

where

$$(B_{ij}, C_{ij}, D_{ij}) = \int_{-c}^{c}(1, z, z^2)E_{ij}\,dz \quad . \tag{4}$$

We agree to write the inverted form of (3) as

$$\varkappa_i = D_{ij}^{-1}M_j + C_{ij}^{-1}N_j \quad , \quad \epsilon_j = C_{ij}^{-1}M_j + B_{ij}^{-1}N_j \quad . \tag{5a, b}$$

Introduction of (5) into equation (2) leads to approximate expressions for face-parallel stresses in terms of stress couples and stress resultants of the form

$$\sigma_i = H_{ij} M_j + F_{ij} N_j \quad , \tag{6}$$

where

$$H_{ij} = (z D_{kj}^{-1} + C_{kj}^{-1}) E_{ik} \quad , \quad F_{ij} = (z C_{kj}^{-1} + B_{kj}^{-1}) E_{ik} \quad . \tag{7}$$

Having equations (6), we use as associated approximate expressions for the transverse stresses τ_λ the equilibrium expressions

$$\tau_1 = -\int_{-c}^{z} (\sigma_{1,1} + \sigma_{3,2}) dz \quad , \quad \tau_2 = -\int_{-c}^{z} (\sigma_{3,1} + \sigma_{2,2}) dz \quad , \tag{8}$$

in conjunction with the conditions of vanishing τ_λ for $z = \pm c$. It is evident from (6) and (8) that, with suitable coefficients K and J, these expressions can be written as

$$\tau_\lambda = K_{\lambda j \mu} M_{j,\mu} + J_{\lambda j \mu} N_{j,\mu} \quad . \tag{9}$$

Having equation (9) we now introduce, in addition to the resultants N_j and the couples M_j, transverse shear resultants Q_λ, and we observe that the set N_j, M_j, Q_λ is subject to the five two-dimensional equilibrium equations

$$M_{1,1} + M_{3,2} = Q_1 \quad , \quad M_{3,1} + M_{2,2} = Q_2 \quad , \quad Q_{1,1} + Q_{2,2} = 0 \quad , \tag{10a}$$

$$N_{1,1} + N_{3,2} = 0 \quad , \quad N_{3,1} + N_{2,2} = 0 \quad . \tag{10b}$$

We may use equations (10) in order to express the τ_λ - instead of in terms of the twelve $M_{j,\mu}$ and $N_{j,\mu}$ - in terms of ten quantities only, these being the two transverse resultants Q_λ, and <u>two</u> sets of <u>four</u> quantities R_Σ and S_Σ defined by

$$R_1 = M_{1,1} - M_{3,2} \ , \ R_2 = M_{1,2} \ , \ R_3 = M_{3,1} - M_{2,2} \ , \ R_4 = M_{2,1} \ , \tag{11a}$$

$$S_1 = N_{1,1} - N_{3,2} \ , \ S_2 = N_{1,2} \ , \ S_3 = N_{3,1} - N_{2,2} \ , \ S_4 = N_{2,1} \ . \tag{11b}$$

The resulting expressions for τ_λ may be written in the form

$$\tau_\lambda = A_{\lambda\mu} Q_\mu + I_{\lambda\Sigma} R_\Sigma + L_{\lambda\Sigma} S_\Sigma \quad , \tag{12}$$

with the problem of the appropriate treatment of the quantities R_Σ and S_Σ being of particular interest in what follows.

Bending and Stretching Uncoupled. It is sufficient for present purposes to limit further discussion to the uncoupled problem of transverse bending, which is given when all the coefficients E_{ij} and $G_{\lambda\mu}$ in (1) are *even* functions of the thickness coordinate z. We obtain the equations for uncoupled transverse bending by setting

$$\epsilon_j = 0 \ , \quad C_{ij} = 0 \ , \quad F_{ij} = 0 \ , \quad J_{\lambda j \mu} = 0 \ , \quad S_{\Sigma} = 0 \ , \tag{13}$$

with the remaining plate problem now consisting of the standard equilibrium equations (10a), in conjunction with approximate two-dimensional constitutive equations and strain displacement relations, with these having to be consistent with the remaining portions of equations (2), (3) and (12).

Use of the Principle of Minimum Complementary Energy. We have as three-dimensional strain energy density expression in terms of stresses, consistent with the stress strain relations (1a, b)

$$W = W_{\sigma} + W_{\tau} = \tfrac{1}{2} E^{-1}_{ij} \sigma_i \sigma_j + \tfrac{1}{2} G^{-1}_{\lambda\mu} \tau_{\lambda} \tau_{\mu} \ . \tag{14}$$

Leaving aside boundary condition considerations, we may obtain a system of approximate two-dimensional constitutive equations by utilizing the variational equation

$$\delta \iint \int_{-c}^{c} W dz \, dx_1 dx_2 = 0 \ , \tag{15}$$

in conjunction with the equilibrium stress expressions

$$\sigma_i = H_{ij} M_j \ , \quad \tau_{\lambda} = K_{\lambda j \mu} M_{j,\mu} \ , \tag{16a, b}$$

subject only to the supplementary two-dimensional equilibrium equation

$$M_{1,11} + 2M_{3,12} + M_{2,22} = 0 \ , \tag{17}$$

which follows from (10a) and which may be introducted into (15) by means of a Lagrangian multiplier function $w(x_1, x_2)$, as done first for the isotropic homogeneous case in [3].

We may alternately use the expression

$$\tau_\lambda = A_{\lambda\mu} Q_\mu + I_{\lambda\Sigma} R_\Sigma \tag{18}$$

for transverse shear where, necessarily,

$$\int_{-c}^{c} A_{\lambda\mu} dz = \delta_{\lambda\mu} \quad , \quad \int_{-c}^{c} I_{\lambda\Sigma} dz = 0 \quad . \tag{19}$$

Introduction of (18) into the associated strain energy portion W_τ then gives further

$$\begin{aligned} \int_{-c}^{c} W_\tau dz &= \tfrac{1}{2}\int_{-c}^{c} G^{-1}_{\lambda\mu}(A_{\lambda\nu} Q_\nu + I_{\lambda\Sigma} R_\Sigma)(A_{\mu\rho} Q_\rho + I_{\mu\Omega} R_\Omega) dz \\ &= \tfrac{1}{2}(X_{\nu\rho} Q_\nu Q_\rho + 2\, Y_{\nu\Omega} Q_\nu R_\Omega + Z_{\Sigma\Omega} R_\Sigma R_\Omega) \quad , \end{aligned} \tag{20}$$

while at the same time

$$\int_{-c}^{c} W_\sigma dz = \tfrac{1}{2}\int_{-c}^{c} E^{-1}_{ij} H_{ik} H_{j\ell} M_k M_\ell \, dz = \tfrac{1}{2} D^{-1}_{ij} M_i M_j \quad . \tag{21}$$

Having equations (15), (20) and (21), in conjunction with the two-dimensional equilibrium equations (10a), we now consider the utilization of this system for the derivation of approximate two-dimensional constitutive equations. In doing this it is of interest to consider a certain sub-class of cases before considering the general problem.

<u>Plates for which $Y_{\nu\Omega} = 0$ and $Z_{\Sigma\Omega} = 0$.</u> For this class of cases we may establish constitutive equations in the same manner as in [4] by introducing the equilibrium equations (10a) into (15) by means of three Lagrange multiplier functions $\varphi_\lambda(x_1, x_2)$ and $w(x_1, x_2)$, so as to have as a variational equation for the derivation of constitutive equations

$$\delta \iint \left[\tfrac{1}{2} D^{-1}_{ij} M_i M_j + \tfrac{1}{2} X_{\lambda\mu} Q_\lambda Q_\mu + Q_{\lambda,\lambda} w \right.$$

$$\left. + (M_{1,1} + M_{3,2} - Q_1)\varphi_1 + (M_{3,1} + M_{2,2} - Q_2)\varphi_2 \right] dx_1 dx_2 = 0 \ . \quad (22)$$

The resulting equations are

$$D^{-1}_{1j} M_j = \varphi_{1,1} \ , \quad D^{-1}_{2j} M_j = \varphi_{2,2} \ , \quad D^{-1}_{3j} M_j = \varphi_{1,2} + \varphi_{2,1} \ , \quad (23)$$

together with

$$X_{1\mu} Q_\mu = w_{,1} + \varphi_1 \ , \quad X_{2\mu} Q_\mu = w_{,2} + \varphi_2 \ . \quad (24)$$

Plates for which some of the $Y_{\nu\Omega}$ and $Z_{\Sigma\Omega}$ are non-vanishing. In considering the variational equation (15) for this general case, we must take account of all or some of the terms with R_Σ in equation (20), in a consistent manner. This may be done without additional Lagrange multipliers by writing the terms R_Σ in (20) in terms of derivatives of the M_i in accordance with equations (11a) and by then using the one constraint equation (17). Alternately, we may introduce equations (11a) as constraint equations, through additional Lagrange multipliers ψ_Σ. If we adopt the latter procedure, we obtain as a generalization of the system (23) and (24) the following set of constitutive equations

$$D^{-1}_{1j} M_j = \varphi_{1,1} + \psi_{1,1} + \psi_{2,2}$$

$$D^{-1}_{2j} M_j = \varphi_{2,2} - \psi_{3,2} + \psi_{4,1} \quad (25)$$

$$D^{-1}_{3j} M_j = \varphi_{1,2} + \varphi_{2,1} - \psi_{1,2} + \psi_{3,1}$$

together with the two relations

$$X_{\nu\rho} Q_\rho + Y_{\nu\Omega} R_\Omega = w_{,\nu} + \varphi_\nu \ , \quad (26)$$

and the four relations

$$Y_{\nu\Omega} Q_\nu + Z_{\Sigma\Omega} R_\Sigma = \psi_\Omega \quad . \tag{27}$$

We note that equations (25) to (27), together with the equilibrium equations (10a) and the defining relations (11a), are a system of sixteen equations for the nine stress measures M_{ij}, Q_ν, R_Σ and seven displacement measures w, φ_ν, ψ_Σ. If we eliminate the quantitites R_Σ and ψ_Σ, we will be left with altogether eight equations for the M_{ij}, Q_ν, w and φ_ν. We refrain from listing these equations and limit ourselves to noting the important fact that the presence of the geometrical variables ψ_Ω in (27) means that it will now not be possible to take account of the effect of transverse shear deformations by means of constitutive equations of a form as simple as the set (24).

A supplementary approximation. It is possible, formally, to use the variational equation (15) for the derivation of approximate constitutive equations without requiring that the transverse shear stress distributions in (18) are in exact three-dimensional equilibrium with the stresses σ_i in (16a) although, in this case, one then no longer uses the principle of minimum complementary energy but rather an ad hoc modification of this principle. One such non-rational modification is to ignore the fact that the quantities R_Σ in (18) are defined in terms of the $M_{i,j}$ in accordance with (11a) and to treat the R_Σ as supplementary free parameter functions. When this is done there is then no occasion for the introduction of the Lagrange multiplier functions ψ_Σ and therewith the constitutive equation system (25) to (27) becomes simplified by the disappearance of all ψ_Σ-terms. This fact is of particular significance insofar as equations (26) and (27) are concerned, because it is then possible to solve (27) in the form $R_\Sigma = -Z^{-1}_{\Sigma\Omega} Y_{\nu\Omega} Q_\nu$ and to write (26) in the wanted form $w_{,\nu} + \varphi_\nu = (X_{\nu\rho} - Y_{\nu\Omega} Z^{-1}_{\Omega\Sigma} Y_{\rho\Sigma}) Q_\rho$.

It appears that the results reported in [2], when specialized to the case of uncoupled stretching and bending, are in fact equivalent to what is obtained in this particular fashion.

Use of a Variational Principle for Stresses and Displacements. Use of the variational equation

$$\delta \iiint \{ u_{1,1}\sigma_1 + u_{2,2}\sigma_2 + (u_{1,2} + u_{2,1})\sigma_3 + (u_{1,z} + u_{z,1})\tau_1$$

$$+ (u_{2,z} + u_{z,2})\tau_2 - W \} \, dz \, dx_1 \, dx_2 = 0 \quad , \qquad (28)$$

in place of equation (15), is known [5] to be consistent, without a priori constraints on stress distributions and with or without independent assumptions concerning displacements. Consequently, if we depart from (28), we may now use the strain energy density expression

$$\int_{-c}^{c} W dz = \tfrac{1}{2}[D_{ij}^{-1} M_i M_j + X_{\nu\rho} Q_\nu Q_\rho + 2 Y_{\nu\Omega} Q_\nu R_\Omega + Z_{\Sigma\Omega} R_\Sigma R_\Omega] \quad , \qquad (29)$$

without considering (11a) as constraint equations for the R_Ω. Furthermore, if we introduce as approximating expressions for displacements, consistent with the stress expressions $\sigma_i = E_{ij} z \varkappa_j$,

$$u_\lambda = z\varphi_\lambda(x_1, x_2) \quad , \quad u_z = w(x_1, x_2) \quad , \qquad (30a, b)$$

we then have in equation (28)

$$\int_{-c}^{c} \{ u_{1,1}\sigma_1 + \ldots + (u_{2,z} + u_{z,2})\tau_2 \} dz = \varphi_{1,1} M_1 + \varphi_{2,2} M_2$$

$$+ (\varphi_{1,2} + \varphi_{2,1}) M_3 + (\varphi_1 + w_{,1}) Q_1 + (\varphi_2 + w_{,2}) Q_2 \quad , \qquad (31)$$

and the Euler equations of the remaining variational problem come out to be the three two-dimensional equilibrium equations (10a), together with constitutive equation of the form (23), in conjunction with an abbreviated system of transverse shear equations

$$X_{\nu\rho}Q_{\rho} + Y_{\nu\Omega}R_{\Omega} = w_{,\nu} + \varphi_{\nu} \quad , \quad Y_{\nu\Omega}Q_{\nu} + Z_{\Sigma\Omega}R_{\Sigma} = 0 \quad , \tag{32a, b}$$

in place of the system (26) and (27).

We note that these results are equivalent to the results in [1], as discussed at the end of the preceding section, and that they are also equivalent to the results in [6], except that in this earlier work the shear stress equations (16b) were taken, with the help of six functions $R^*_{j\mu}$, in the form

$$\tau_{\lambda} = K_{\lambda j\mu} R^*_{j\mu} \quad , \tag{33}$$

in place of equations (18), with the Q_{ν} in (31) following from (33) in the form

$$Q_{\nu} = \left(\int_{-c}^{c} K_{\nu j\mu}\, dz\right) R^*_{j\mu} = C_{\nu j\mu} R^*_{j\mu} \quad . \tag{34}$$

There occur, in connection with the use of (33) in place of (18) as approximation for the τ_{λ}, inconveniences in regard to the appearance of singular matrices†. While it is possible to deal with these singularities as they occur in specific cases, without affecting the validity of the results, it is nevertheless preferable to isolate the effect of the Q_{μ}-terms, as in equations (18).

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† The present author is indebted to Dr. Cohen for bringing this fact to his attention.

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